

Entropy of quantum-corrected black holes

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The approximate renormalized one-loop effective action of the quantized massive scalar, spinor and vector field in a large mass limit, i.e., the lowest order of the DeWitt-Schwinger expansion involves the coincidence limit of the Hadamard-DeWitt coefficient a_3 . Building on this and using Wald's approach we shall construct the general expression describing entropy of the spherically-symmetric static black hole being the solution of the semi-classical field equations. For the concrete case of the quantum-corrected Reissner-Nordström black hole this result coincides, as expected, with the entropy obtained by integration of the first law of black hole thermodynamics with a suitable choice of the integration constant. The case of the extremal quantum corrected black hole is briefly considered.

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I. INTRODUCTION

Mathematical difficulties encountered in the attempts to construct the renormalized stress-energy tensor and other characteristics of the quantized fields in curved background are well known, and, except extremely simple cases, they invalidate exact treatment of the semi-classical Einstein field equations. On the other hand, the back reaction programme, as it is understood today, requires knowledge of the functional dependence of the stress-energy tensor on a wide class of metrics. Treating the renormalized stress-energy tensor of the quantized field as a source term of the semi-classical equations, one can, in principle, construct the self-consistent solution to the equations and analyze evolution of the system unless the quantum gravity effects become dominant. It is natural, therefore, that in order to make the back reaction calculations tractable, one has to refer to some approximations or even try numerical techniques.

Nowadays, the literature devoted to calculation of $\langle T_a^b \rangle$ of the quantized fields in the spacetimes of black holes is vast indeed. In the Schwarzschild geometry we have good understanding of the stress-energy tensor of the quantized massive and massless fields in the Boulware, Unruh and Hartle-Hawking states. Specifically, due to excellent numerical work we have results that may be considered as exact [1, 2, 3, 4, 5, 6, 7, 8]. On the other hand, analytical [9, 10, 11, 12] and semi-analytical [13, 14, 15, 16, 17, 18] approximations have been constructed and successfully applied in numerous physically interesting cases. Moreover, at the expense of increasing number of numerical data required to construct the stress-energy tensor, a few best-fit models has been proposed [19, 20, 21].

Unfortunately, less is known of the observables in other geometries. There is, however, a remarkable exception: It has been shown that for sufficiently massive fields (i.e. when the Compton length is much smaller than the characteristic radius of curvature, where the latter means any characteristic length scale of the geometry) the asymptotic expansion of the effective action in powers of m^{-2} may be used. Here m is a mass of the quantized field. It is because the nonlocal contribution to the total effective action can be neglected and the vacuum polarization part is local and determined by the geometry of the spacetime in question. This is a very fortunate feature as it allows a straightforward calculation of the approximate stress-energy tensor simply by employing the standard relation

$$\frac{2}{g^{1/2}} \frac{\delta}{\delta g^{ab}} S_q^{(s)} = -\langle T_{ab}^{(s)} \rangle. \quad (1)$$

One expects, by construction, that the result satisfactorily approximates exact T^{ab} in any geometry, provided the temporal changes of spacetime are small and the mass of the quantized field is sufficiently large. Such a tensor of the massive scalar, spinor and vector fields specialized to the Ricci flat geometries has been calculated by Frolov and Zel'nikov [22, 23, 24]. On the other hand, within the framework of the sixth-order WKB approximation, $\langle T^{ab} \rangle_{ren}$ of the massive scalar field propagating in a general spherically-symmetric spacetime has been constructed by Anderson,

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Hiscock and Samuel [7]. (Slightly different method has been adopted in [25]). The Frolov and Zel'nikov results have been subsequently extended to more general geometries in [26, 27].

The stress-energy tensor of the quantized massive fields in a large mass limit has been used in a number of physically interesting cases. Among them a prominent role is played by the back reaction of the quantized field upon black hole spacetimes. Such quantum corrected solutions have been studied from various points of view in Refs. [28, 29, 30, 31, 32], where the corrections to the location of the event horizon, mass and temperature as well as important issue of extreme black holes have been considered. On the other hand, less is known about the entropy of such systems (see, however, Ref. [33]).

At one loop the black hole entropy is a sum

$$\mathcal{S}_{BH} = \mathcal{S}_{cl}[g_{ab}] + \mathcal{S}_q[g_{ab}], \quad (2)$$

where $\mathcal{S}_{cl}[g_{ab}]$ is the entropy obtained from the gravitational action whereas $\mathcal{S}_q[g_{ab}]$ is to be constructed from the one loop effective action and g_{ab} is the metric tensor of the quantum corrected black hole geometry, actually being a self consistent solution to the semi classical Einstein field equations. Unfortunately, complexity of the field equations invalidates, even in simplest cases, construction of the exact solutions. However, the nature of the contribution of the quantized fields to the total action suggests the line of attack. Imposing appropriate boundary conditions one can easily devise the perturbative approach to the problem. For example, for the spherically-symmetric and electrically charged static black hole one can choose the *exact* location of the event horizon, r_+ , and the charge, e , to be related with the integration constants. Consequently, the zeroth-order solution coincides with the Reissner- Nordström geometry characterized by the exact r_+ and e .

Since the stress-energy tensor of the quantized field in a large mass limit describes the vacuum polarization effects rather than the real excitations (the latter are exponentially small) one can forget about its material origin and treat the one loop effective action as an additional higher curvature (derivative) correction to the total gravitational action. As we shall see below it is a very fortunate feature of the theory.

One can evaluate the entropy of the quantum corrected black hole by integrating of the first law of thermodynamics. On the other hand one can employ the Noether charge technique [34, 35, 36, 37, 38] – a method which seems to be best suited not only in calculations within the framework of the higher derivative gravity and the low-energy limit of the string theory but also in the calculations employing the purely local one-loop effective action of the quantized massive fields. Technically speaking, in this approach, one has to calculate derivatives of the effective Lagrangian with respect to the Riemann tensor and its (symmetrized) covariant derivatives and integrate the result over a two-dimensional section of the event horizon.

In this paper using Wald's approach we shall construct the general expression describing entropy of the spherically-symmetric static black hole being the solution of the semi-classical field equations. Obtained formula can be applied in any black hole spacetime provided the assumptions necessary to construct the one loop effective action are satisfied and the action of the classical matter fields does not functionally depend on the Riemann tensor and its covariant derivatives. The structure of the effective action suggests that the entropy, except the standard Bekenstein-Hawking term, contains also the higher curvature contribution [39, 40, 41, 42]. In general, there will be terms coming from the renormalized quadratic action as well as the contribution of the quantized fields.

For the quantum corrected Reissner-Nordström black hole we shall demonstrate that identical result, up to an integration constant, can be obtained from the first law of black hole thermodynamics. It should be observed that when using Wald's prescription, the model considered in this paper requires the zeroth- order solution only, whereas integration of the first law requires also its first-order corrections.

Although the quantized massive fields propagating in the black hole geometry has been analyzed in numerous papers (see for example [40]), their contribution to the entropy has been ignored. Indeed, after absorbing the divergent part of the DeWitt-Schwinger action into the quadratic Lagrangian in the process of renormalization of the bare coupling constants, the higher order terms have been neglected. It is justified whenever we have no interest in the entropy of the quantized field itself and the quantum-corrected black hole, i. e., if we restrict ourselves to the effectively quadratic gravity, but otherwise is unsatisfactory. To the best knowledge of the author it is a first attempt to calculate the entropy of such systems in a more general setting than the Schwarzschild solution [33].

II. EFFECTIVE ACTION

In this paper we shall restrict ourselves to the massive scalar, spinor and vector fields satisfying the equations (conventions are $R_{ab} = R^c_{acb} \sim \partial_c \Gamma^c_{ab}$, signature $- , + , + , +$)

$$(-\nabla_a \nabla^a + \xi R + m^2)\phi^{(0)} = 0, \quad (3)$$

$$(\gamma^a \nabla_a + m)\phi^{(1/2)} = 0, \quad (4)$$

$$(\delta_b^a \nabla_c \nabla^c - \nabla_b \nabla^a - R_b^a - \delta_b^a m^2)\phi^{(1)} = 0, \quad (5)$$

respectively. Here ξ is the curvature coupling constant, and γ^a are the Dirac matrices obeying standard relations $\gamma^a \gamma^b + \gamma^b \gamma^a = 2g^{ab}\hat{1}$. The first order term of the renormalized effective action of the quantized scalar, spinor and vector fields in a large mass limit is constructed from the (traced) coincidence limit of the fourth Hadmard-DeWitt coefficients $a_3^{(s)}$, and can be written as [43, 44, 45]

$$S_q^{(s)} = \frac{1}{32\pi^2 m^2} \int d^4x g^{1/2} \left\{ \begin{array}{l} [a_3^{(0)}] \\ -tr[a_3^{(1/2)}] \\ tr[a_3^{(1)}] - [a_3^{(0)}]_{\xi=0} \end{array} \right\} \quad (6)$$

The coefficients $a_0^{(s)}$, $a_1^{(s)}$ and $a_2^{(s)}$ contribute to the divergent part of the action,

$$S_{div} = \int d^4x g^{1/2} \left(\Lambda_B + \frac{1}{16\pi G_B} R + \alpha_B R^2 + \beta_B R_{ab} R^{ab} + \gamma_B R_{abcd} R^{abcd} \right), \quad (7)$$

where the subscript B indicates that the constants are bare, and have to be absorbed by the quadratic gravitational action. Henceforth, the renormalized Newton constant, G , is set to 1.

The total action of the system is

$$S = S_G + S_{sq} + S_m + S_q^{(s)} = \int d^4x g^{1/2} \mathcal{L}, \quad (8)$$

where S_G is the Einstein-Hilbert action

$$S_G = \frac{1}{16\pi} \int d^4x g^{1/2} R, \quad (9)$$

S_{sq} is the renormalized quadratic action

$$S_{sq} = \int d^4x g^{1/2} (\Lambda + \alpha R^2 + \beta R_{ab} R^{ab} + \gamma R_{abcd} R^{abcd}), \quad (10)$$

S_m is the action of the classical matter, and finally $S_q^{(s)}$ denote the action of the quantized massive fields.

Although our primary aim is to construct the general expression describing the entropy of the static and spherically symmetric quantum corrected black holes, we use obtained results in a spacetime of the electrically charged black hole. Here we shall restrict ourselves to the simplest model with

$$S_m = S_{em} = -\frac{1}{16\pi} \int d^4x g^{1/2} F_{ab} F^{ab}, \quad (11)$$

where $F_{ab} = \nabla_a A_b - \nabla_b A_a$ and A_a is the electromagnetic potential. In doing so we shall ignore other (possible) higher order terms such as $(F_{ab} F^{ab})^2$ and $R_{ab} F^{ac} F_c^b$.

Let us return to the effective action of the quantized fields. It has been demonstrated that the approximate one-loop effective action in a large mass limit is given by [43, 44, 45]

$$\begin{aligned} S_q^{(s)} = \frac{1}{192\pi^2 m^2} \int d^4x g^{1/2} & \left(\alpha_1^{(s)} R \nabla_a \nabla^a R + \alpha_2^{(s)} R_{ab} \nabla_c \nabla^c R^{ab} + \alpha_3^{(s)} R^3 \right. \\ & + \alpha_4^{(s)} R R_{ab} R^{ab} + \alpha_5^{(s)} R R_{abcd} R^{abcd} + \alpha_6^{(s)} R_b^a R_c^b R_a^c + \alpha_7^{(s)} R^{ab} R_{cd} R_a^c R_b^d \\ & \left. + \alpha_8^{(s)} R_{ab} R_{cde}^a R^{bcde} + \alpha_9^{(s)} R_{ab}^{cd} R_{cd}^{ef} R_{ef}^{ab} + \alpha_{10}^{(s)} R_a^c R_b^d R_e^a R_f^b R_c^e R_d^f \right), \end{aligned} \quad (12)$$

where the numerical coefficients $\alpha_i^{(s)}$ depending on the spin of the massive field are tabulated in Table I. It should be emphasized that the status of the renormalized constants α , β and γ on the one hand and coefficients $\alpha_i^{(s)}$ on the other is different. Indeed, the former should be measured empirically, whereas the latter are unambiguously determined by the spin of the field. All we can say at the moment is that they are extremely small since otherwise they would give rise to observational effects. Henceforth, for simplicity, we shall equate them to zero and only briefly discuss their contribution to the entropy in Sec IV A.

TABLE I: The coefficients $\alpha_i^{(s)}$ for the massive scalar, spinor, and vector field

	s = 0	s = 1/2	s = 1
$\alpha_1^{(s)}$	$\frac{1}{2}\xi^2 - \frac{1}{5}\xi + \frac{1}{56}$	$-\frac{3}{140}$	$-\frac{27}{280}$
$\alpha_2^{(s)}$	$\frac{1}{140}$	$\frac{1}{14}$	$\frac{9}{28}$
$\alpha_3^{(s)}$	$(\frac{1}{6} - \xi)^3$	$\frac{1}{432}$	$-\frac{5}{72}$
$\alpha_4^{(s)}$	$-\frac{1}{30}(\frac{1}{6} - \xi)$	$-\frac{1}{90}$	$\frac{31}{60}$
$\alpha_5^{(s)}$	$\frac{1}{30}(\frac{1}{6} - \xi)$	$-\frac{7}{720}$	$-\frac{1}{10}$
$\alpha_6^{(s)}$	$-\frac{8}{945}$	$-\frac{25}{376}$	$-\frac{52}{63}$
$\alpha_7^{(s)}$	$\frac{2}{315}$	$\frac{47}{630}$	$-\frac{19}{105}$
$\alpha_8^{(s)}$	$\frac{1}{1260}$	$\frac{19}{630}$	$\frac{61}{140}$
$\alpha_9^{(s)}$	$\frac{17}{7560}$	$\frac{29}{3780}$	$-\frac{67}{2520}$
$\alpha_{10}^{(s)}$	$-\frac{1}{270}$	$-\frac{1}{54}$	$\frac{1}{18}$

III. THE QUANTUM CORRECTED REISSNER-NORDSTRÖM BLACK HOLE

Setting the renormalized constants to zero and differentiating functionally S with respect to the metric tensor one obtains the semi-classical field equations in their simplest form:

$$G_{ab} = 8\pi \left(T_{ab}^{(m)} + \langle T_{ab}^{(s)} \rangle \right), \quad (13)$$

where $T_{ab}^{(m)}$ and $\langle T_{ab}^{(s)} \rangle$ are the classical and quantum part of the stress-energy tensor, respectively. In this section we shall briefly discuss the perturbative solution to the semi-classical Einstein field equations describing spherically-symmetric and electrically charged static black hole. Although some of the results presented in this section are not new: the massive scalars have been considered in Refs. [28, 31], whereas solutions for the spinor and vector fields have been constructed in [31], we shall repeat, for readers' convenience, the main points of their derivation. In doing so we shall display most of the final results in a more general form than it was done in [28, 31].

The stress-energy tensor of the massive scalar field with arbitrary curvature coupling has been calculated using two different methods. The calculations presented in Ref. [7] were based on the sixth-order WKB approximation of the solutions of the radial scalar field equation and summation thus obtained mode functions by means of the Abel-Plana formula. On the other hand, to construct the stress-energy tensor of Ref. [26, 27] one has to functionally differentiate the effective action with respect to the metric tensor. The equality of the final results is not surprising as there is a one-to-one correspondence between the order of the WKB approximation and the order of DeWitt-Schwinger expansion.

The approximation of the stress-energy tensor considered in this paper is increasingly accurate as the ratio λ_C/L approaches zero, where λ_C is the Compton length of the field whereas L is the characteristic radius of curvature of the black hole geometry, i. e. when $mM \gg 1$, where M is the black hole mass. Detailed numerical calculation carried out by Taylor et al. [7, 28] in the Reissner-Nordström background shows that there are a good agreement between the numerical results and the approximate $\langle T_{ab}^{(s)} \rangle$. For example, for $mM \geq 2$ the deviation of the approximate stress-energy tensor from the exact one lies within a few percent.

Since the terms constructed from R^2 , $R_{ab}R^{ab}$ and the Kretschmann scalar are absent in the renormalized action, we will ignore effects of the quadratic gravity. Their influence upon electrically charged black hole has been extensively studied in a number of papers (see for example [46, 47, 48, 49] and the references cited therein) and appropriate effects can easily be incorporated into the final result.

As is well known the spherically-symmetric and static configuration can be described by a general line element of the form

$$ds^2 = -e^{2\psi(r)} f(r) dt^2 + f^{-1}(r) dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\phi^2), \quad (14)$$

where $f(r)$ and $\psi(r)$ are two unknown functions. Now, let us look more closely at each of the terms in Eq. (13). First, observe that making use of the definition, after some algebra, one obtains the stress-energy tensor of the quantized massive field that consists of approximately 100 terms constructed from the curvature, its contractions and covariant derivatives. As the final result of the calculations is rather lengthy we shall not display it here and refer interested reader to [26, 27]. On the other hand, the classical part of the total stress-energy tensor, identified here with the electromagnetic stress-energy tensor, $T_a^{(em)b}$, and compatible with the assumed symmetry is simply

$$T_t^{(em)t} = T_r^{(em)r} = -T_\theta^{(em)\theta} = -T_\phi^{(em)\phi} = -\frac{C_1^2}{8\pi r^4}, \quad (15)$$

where C_1 is to be identified with the electric charge, e .

Even with the simplifying substitution

$$f(r) = 1 - \frac{2M(r)}{r}, \quad (16)$$

the equations of motion constructed for a line element (14) are too complicated to be solved exactly. Fortunately, one can easily devise the perturbative approach to the problem, treating the higher derivative terms (one loop effective action) as small perturbations. Such a procedure also guarantees exclusion of the spurious solutions which are likely to appear as the resulting equations involve sixth-order derivatives of the unknown functions $M(r)$ and $\psi(r)$.

Now, in order to simplify calculations and to keep control of the order of terms in complicated series expansions, we shall introduce another (dimensionless) parameter ε , substituting $\alpha_i^{(s)} \rightarrow \varepsilon \alpha_i^{(s)}$. We shall put $\varepsilon = 1$ in the final stage of calculations. For the unknown functions $M(r)$ and $\psi(r)$ we assume that they can be expanded as

$$M(r) = \sum_{i=0}^m \varepsilon^i M_i(r) + O(\varepsilon^{m+1}) \quad (17)$$

and

$$\psi(r) = \sum_{i=1}^m \varepsilon^i \psi_i(r) + O(\varepsilon^{m+1}). \quad (18)$$

The system of differential equations for $M_i(r)$ and $\psi_i(r)$ is to be supplemented with the appropriate, physically motivated boundary conditions. First, it seems natural to demand

$$M(r_+) = \frac{r_+^{d-1}}{2}, \quad (19)$$

or, equivalently, $M_0(r_+) = r_+^{d-1}/2$ and $M_i(r_+) = 0$ for $i \geq 1$, where r_+ denotes the exact location of the event horizon. Such a choice leads naturally to the horizon defined mass. On the other hand, one can use the total mass of the system as seen by a distant observer

$$\mathcal{M} = M(\infty). \quad (20)$$

For the function $\psi(r)$ we shall always adopt the natural condition $\psi(\infty) = 0$. Since the results obtained for each set of boundary conditions are not independent, one can easily transform solution of the first type into the solution of the second type (and vice versa). In the course of the calculations one can safely use each of them and the particular choice of representation is dictated by its usefulness.

It can easily be shown that the solution parametrized by the exact location of the event horizon of the quantum-corrected black hole, r_+ , and the electric charge, e , can be written as

$$f(r) = 1 - \frac{r_+}{r} + \frac{e^2}{r^2} - \frac{e^2}{rr_+} + \frac{8\pi\varepsilon}{r} \int_{r_+}^r dr' r'^2 \langle T_t^{(s)t} \rangle \quad (21)$$

and

$$\psi(r) = 4\pi\varepsilon \int_{\infty}^r r' \left(\langle T_r^{(s)r} \rangle - \langle T_t^{(s)t} \rangle \right) \left(1 - \frac{2M(r')}{r'} \right)^{-1} dr'. \quad (22)$$

The zeroth-order line element is obtained by putting (formally) $\varepsilon = 0$ in $f(r)$ and $\psi(r)$. It coincides with the Reissner-Nordström solution, as expected.

Before we proceed further, let us observe that the difference between the (rr) and (tt) components of the stress-energy tensor factorizes as

$$\langle T_r^{(s)r} \rangle - \langle T_t^{(s)t} \rangle = \left(1 - \frac{r_+}{r} + \frac{e^2}{r^2} - \frac{e^2}{rr_+} \right) F^{(s)}(r), \quad (23)$$

where $F^{(s)}(r)$ is a regular function, and, consequently, the integral (22) simplifies to

$$\psi(r) = \varepsilon \psi_{(1)} = 4\pi\varepsilon \int_{\infty}^r F^{(s)}(r') r' dr'. \quad (24)$$

Now, inserting the zeroth-order line element into the stress-energy tensor and performing the necessary integration one obtains the desired solution of the semiclassical equations. The general solution valid for any set of numerical coefficients $\alpha_i^{(s)}$ is too lengthy to be displayed here. Thus, we shall collect the concrete form of functions $f(r)$ and $\psi(r)$ calculated for the scalar, spinor and vector field in Appendix.

It is possible to express the functions f and ψ in a more familiar form by introducing the horizon defined mass M_H , i. e., to represent the solution in terms of (e, M_H) rather than (e, r_+) or (e, \mathcal{M}) . This can be easily done employing the equality

$$M_H = \frac{r_+}{2} + \frac{e^2}{2r_+}, \quad (25)$$

and with such a choice of the representation the exact location of the event horizon is related to the horizon defined mass by the classical formula

$$r_+ = M_H + (M_H^2 - e^2)^{1/2}. \quad (26)$$

IV. ENTROPY OF THE QUANTUM CORRECTED BLACK HOLE

A. Noether charge technique

For the Lagrangian involving the Riemann tensor and its symmetric derivatives up some finite order n , the Wald's Noether charge entropy may be compactly written in the form [35, 36]

$$\mathcal{S} = -2\pi \int d^2x (h)^{1/2} \sum_{m=0}^n (-1)^m \nabla_{(e_1 \dots e_m)} Z^{e_1 \dots e_m; abcd} \epsilon_{ab} \epsilon_{cd} \quad (27)$$

where

$$Z^{e_1 \dots e_m; abcd} = \frac{\partial \mathcal{L}}{\partial \nabla_{(e_1 \dots e_m)} R_{abcd}}, \quad (28)$$

h is the determinant of the induced metric, ϵ_{ab} is the binormal to the bifurcation sphere, and the integration is carried out across the bifurcation surface. Actually \mathcal{S} can be evaluated not only on the bifurcation surface but on an arbitrary cross-section of the Killing horizon. Since $\epsilon_{ab}\epsilon_{cd} = \hat{g}_{ac}\hat{g}_{bd} - \hat{g}_{ad}\hat{g}_{bc}$, where \hat{g}_{ac} is the metric in the subspace normal to cross section on which the entropy is calculated, one can rewrite Eq. (27) in the form

$$\mathcal{S} = 4\pi \int d^2x h^{1/2} \sum_{m=0}^n (-1)^m \nabla_{(e_1 \dots e_m)} Z^{e_1 \dots e_m; abcd} \hat{g}_{ac} \hat{g}_{bd}. \quad (29)$$

The tensor \hat{g}_{ab} is related to $V^a = K^a/||K||$ (K^a is the timelike Killing vector) and the unit normal n^a by the formula $\hat{g}_{ab} = V_a V_b + n_a n_b$.

Inspection of the total action functional shows that except S_m all the terms in the right hand side of Eq. (8) contribute to the entropy. Moreover, as the one-loop effective Lagrangian involves two terms which are constructed from the second covariant derivatives of the contractions of R_{bcd}^a with respect to the metric tensor, one has to take $n = 2$ in Eq. (27). This, of course, leads to additional computational complications.

After some algebra, for a general static and spherically symmetric black hole, one has

$$\begin{aligned} \mathcal{S} = & \pi r_+^2 + \frac{\varepsilon r_+^2}{12m^2} \left\{ 2\alpha_1^{(s)} \nabla_a \nabla^a R + \alpha_2^{(s)} (\nabla_a \nabla^a R_t^t + \nabla_a \nabla^a R_r^r) + 3\alpha_3^{(s)} R^2 \right. \\ & + \alpha_4^{(s)} [R (R_t^t + R_r^r) + R_{ab} R^{ab}] + \alpha_5^{(s)} (R_{abcd} R^{abcd} + 4R_{tr}^{tr}) \\ & + \frac{3}{2}\alpha_6^{(s)} [(R_t^t)^2 + (R_r^r)^2] + \alpha_7^{(s)} [R_{tr}^{tr} (R_t^t + R_r^r) + R_t^t R_r^r + 2R_\theta^\theta (F_t + F_r)] \\ & \left. + 2\alpha_8^{(s)} [(R_{tr}^{tr})^2 + F_t^2 + F_r^2 + R_{tr}^{tr} (R_t^t + R_r^r)] + 12\alpha_9^{(s)} (R_{tr}^{tr})^2 + 6\alpha_{10}^{(s)} F_t F_r \right\}_{|r_+}, \end{aligned} \quad (30)$$

where

$$F_t = R_{t\theta}^{t\theta} = R_{t\phi}^{t\phi} \quad (31)$$

and

$$F_r = R_{r\theta}{}^{r\theta} = R_{r\phi}{}^{r\phi}. \quad (32)$$

Note that it is a quite general result and it can be used as long as the black hole geometry is spherically-symmetric, the stress-energy tensor of the classical fields is independent of the Riemann tensor and $\lambda_C/L \ll 1$. Similarly, the contribution of the quadratic part of the action to the entropy, mostly ignored in this paper, is given by

$$\mathcal{S}_2 = 32\pi^2 r_+^2 \left[\alpha R + \frac{1}{2}\beta (R_t^t + R_r^r) + 2\gamma R_{tr}{}^{tr} \right]_{|r_+} \quad (33)$$

Now, we are in position to employ the general formula describing the entropy of the quantum corrected black hole in the concrete case of the Reissner-Nordström geometry. Inspection of Eq. (30) shows that it suffices to retain only the zeroth-order solution. It is because the first term in the right hand side of Eq. (30) depends solely on the radius of the event horizon, which is considered as the exact quantity here. Consequently, inclusion of the first-order terms in the line element term would give rise to $O(\varepsilon^2)$ terms in the final result. Simple calculations yield

$$\begin{aligned} \mathcal{S} = & \pi r_+^2 - \frac{\varepsilon}{m^2 r_+^6} \left[\frac{2}{3} e^2 (e^2 - r_+^2) \alpha_2^{(s)} - \frac{1}{3} e^4 \alpha_4^{(s)} - \frac{1}{3} (3r_+^4 - 6e^2 r_+^2 + 5e^4) \alpha_5^{(s)} \right. \\ & - \frac{1}{4} e^4 \alpha_6^{(s)} - \frac{1}{12} e^2 (4r_+^2 - 7e^2) \alpha_7^{(s)} - \frac{1}{12} (14e^2 r_+^2 - 3r_+^4 - 17e^4) \alpha_8^{(s)} \\ & \left. - (r_+^2 - 2e^2)^2 \alpha_9^{(s)} + \frac{1}{8} (e - r_+)^2 (e + r_+)^2 \alpha_{10}^{(s)} \right] \equiv \pi r_+^2 + \Delta S^{(s)}. \end{aligned} \quad (34)$$

Substituting tabulated values of the coefficients $\alpha_i^{(s)}$ into the above equation, one obtains for the scalar, spinor and vector fields

$$\Delta S^{(0)} = \frac{1}{7560 m^2 r_+^2} (15r_+^4 + 504r_+^2 e^2 \eta - 48r_+^2 e^2 - 336e^4 \eta + 49e^4 - 252r_+^4 \eta), \quad (35)$$

$$\Delta S^{(1/2)} = \frac{1}{5040 m^2 r_+^2} (77e^4 - 48r_+^2 e^2 + 8r_+^4) \quad (36)$$

and

$$\Delta S^{(1)} = \frac{1}{2520 m^2 r_+^2} (148r_+^2 e^2 - 7e^4 - 27r_+^4), \quad (37)$$

where $\eta = \xi - 1/6$. Note, that depending on the values of $q = |e|/r_+$ and the curvature coupling constant ξ , the contribution of $\Delta S^{(s)}$ can be negative but the total entropy of the system is, of course, always positive. For example, for vanishing electric charge $\Delta S^{(s)}$ is always positive for the spinor and negative for the vector fields. On the other hand, the sign of the contribution of the scalar field is negative for $\xi > 19/84$. Thus, it is positive for the conformal and the minimal coupling.

The renormalized action of quadratic gravity (10) leads to

$$\mathcal{S}_2 = \pi r_+^2 - 32\pi^2 \beta \frac{e^2}{r_+^2} + 64\pi^2 \gamma \left(1 - 2 \frac{e^2}{r_+^2} \right). \quad (38)$$

The higher order terms in \mathcal{S}_2 are 4π times that of Ref [40], as expected. Since the Gauss-Bonnet invariant

$$S_{GB} = \int d^4 x g^{1/2} (R^2 - 4R_{ab}R^{ab} + R_{abcd}R^{abcd}) \quad (39)$$

has zero functional derivative with respect to the metric tensor, the Kretschmann scalar can be relegated from the action (10). Making use of the easy-to-prove identity

$$r_+^2 \left(\frac{1}{2} R - R_t^t - R_r^r + R_{tr}{}^{tr} \right)_{|r_+} = 1 \quad (40)$$

valid for the zeroth-order line element (14) with (16-18), one concludes that the contribution of the S_{GB} to the entropy is a constant independent of r_+ .

Putting $e = 0$ in Eqs. (35-37) one obtains the entropy of the quantum corrected Schwarzschild black hole. The entropy of such a black hole confined in a spherical box of a radius R_0 has been constructed in Ref. [33]. It could be shown that for $R_0 \rightarrow \infty$ the results of [33] coincide with $e = 0$ limit of Eqs. (35-37).

B. First law

Since the calculations of the previous subsection are rather complicated, it is reasonable to rederive the results (34-37) using different approach. Here we shall demonstrate that technically independent calculation of the entropy can be carried out employing the first law of thermodynamics

$$M = TdS + \sum_i \mu_i dQ_i, \quad (41)$$

where T is the temperature and μ_i are the chemical potentials corresponding to the conserved charges Q_i . Making use of Eq. (41) one has

$$\begin{aligned} S &= \int T^{-1} d\mathcal{M} + \mathcal{S}_0 \\ &= \int T^{-1} \left(\frac{\partial \mathcal{M}}{\partial r_+} \right)_{Q_i} dr_+ + \mathcal{S}_0. \end{aligned} \quad (42)$$

The integration constant \mathcal{S}_0 does not depend on r_+ , but possibly depends on the coupling constants.

In the present approach it is necessary to retain in the line element all the terms proportional to ε . Specifically, to construct the entropy one has to know to the required order both \mathcal{M} , i. e., the total mass of the system as seen by a distant observer and the temperature. The former quantity may be calculated from the definition (20) and the formulas collected in Appendix, whereas the temperature can be constructed using the Euclidean form of the line element obtained from the Wick rotation ($t \rightarrow -it$). Now, the geometry has no conical singularity as $r \rightarrow r_+$, provided the ‘time’ coordinate is periodic with a period β given by

$$\beta = 4\pi \lim_{r \rightarrow r_+} (g_{tt} g_{rr})^{1/2} \left(\frac{d}{dr} g_{tt} \right)^{-1}. \quad (43)$$

The Hawking temperature, T_H , is related to β by means of the standard formula

$$\beta = \frac{1}{T_H}. \quad (44)$$

It can be demonstrated that the total mass $\mathcal{M}^{(s)}$ calculated from (20) is given by [28, 31]

$$\begin{aligned} \mathcal{M}^{(s)} &= M_H + \frac{\varepsilon}{\pi m^2 r_+^9} \left\{ \frac{\alpha_2^{(s)}}{1512} e^2 (217e^4 + 252r_+^4 - 459e^2 r_+^2) \right. \\ &\quad + (r_+^2 - e^2) \left[\frac{\alpha_4^{(s)}}{12} e^4 + \frac{\alpha_5^{(s)}}{12} (3r_+^4 - 6e^2 r_+^2 + 5e^2) + \frac{\alpha_6^{(s)}}{16} e^4 \right] \\ &\quad + \frac{\alpha_7^{(s)}}{216} e^2 (6r_+^2 - 7e^2) (3r_+^2 - 4e^2) + \frac{\alpha_8^{(s)}}{432} (27r_+^6 + 261e^4 r_+^2 - 153e^2 r_+^4 - 139e^6) \\ &\quad + \frac{\alpha_9^{(s)}}{504} (105r_+^6 + 909e^4 r_+^2 - 585e^2 r_+^4 - 445e^6) \\ &\quad \left. + \frac{\alpha_{10}^{(s)}}{2016} (21r_+^6 + 117e^4 r_+^2 - 99e^2 r_+^4 - 43e^6) \right\}. \end{aligned} \quad (45)$$

Further, restricting the general expression (43) to the line element (14) one obtains

$$T_H = \frac{1}{4\pi} e^{\psi(r_+)} \frac{df}{dr} \Big|_{r=r_+}, \quad (46)$$

and consequently for the quantum-corrected Reissner-Nordström black hole the final result is given by

$$\begin{aligned}
T_H = & \frac{1}{4\pi r_+} \left(1 - \frac{e^2}{r_+^2} \right) + \frac{\varepsilon}{\pi^2 m^2 r_+^{11}} \left\{ \frac{\alpha_2^{(s)}}{48} (r_+^2 - e^2) (7e^2 - 4r_+^2) \right. \\
& - (r_+^2 - 3e^2) \left[\frac{\alpha_4^{(s)}}{24} e^4 + \frac{\alpha_5^{(s)}}{24} (3r_+^4 - 6e^2 r_+^2 + 5e^2) + \frac{\alpha_6^{(s)}}{32} e^4 \right] \\
& + \frac{\alpha_7^{(s)}}{48} e^2 (2r_+^4 + 7e^4 - 6e^2 r_+^2) - \frac{\alpha_8^{(s)}}{96} (3r_+^6 + 45e^4 r_+^2 - 23e^2 r_+^4 - 37e^6) \\
& \left. - \frac{\alpha_9^{(s)}}{112} (7r_+^6 + 147e^4 r_+^2 - 73e^2 r_+^4 - 109e^6) + \frac{\alpha_{10}^{(s)}}{448} (r_+^2 - e^2) (7r_+^4 - 8e^2 r_+^2 - e^6) \right\}.
\end{aligned} \tag{47}$$

Now we are in position to calculate the entropy. Substituting (47) into (42), expanding and collecting the terms with the like powers of ε , and, finally, linearizing the thus obtained result, after some algebra, one gets

$$\mathcal{S}_T = \mathcal{S} + \mathcal{S}_0 \tag{48}$$

where \mathcal{S} is given by (34) and \mathcal{S}_0 is the integration constant. Thus, \mathcal{S}_T coincides with the entropy calculated within the Wald approach provided $\mathcal{S}_0 = 0$. Technically speaking, both calculations are quite different and the identity of the results may be considered as an important consistency check.

C. The $r_+ = r_-$ limit

The issue of the entropy of the extreme black hole has been a subject of long-standing debate. In general, there are two main lines of reasoning, yielding, unfortunately, different results. The first one, originated in [50], consists in the observation that the Euclidean topologies of extreme and nonextreme black holes are different. This fact has profound consequences, the most important of which is the observation that the entropy of the extreme black holes does not obey area law. Actually, the authors of Ref. [50] argued that it is zero, although one can invent modification of the method adopted in [50] to draw quite the opposite conclusion [51]. This behaviour can be described as extremalization after quantization as opposed to the approach in which the order of the operations is reversed.

On the other hand, there is still growing evidence, that the entropy of the extreme black holes, at least for some classes of them, should obey the area law plus (possible) additional theory-dependent terms. For example, in the influential paper [52], this result has been shown by counting microstates of the certain class of black holes in string theory. Moreover, it has been explicitly demonstrated (see for example [53, 54]) that the macroscopic entropy calculated with the aid of the Wald's prescription is in agreement with the entropy obtained by counting microstates for extremal black holes considered in Refs. [55, 56].

In the following we shall assume that the entropy formula can be extrapolated to the case of the extremal black holes. However, even if it turns out to be wrong and the entropy is discontinuous, such calculations make sense. It should be noted that adjusting the set of parameters suitably one can approach the extremal configuration arbitrarily close. The entropy of such configurations can be calculated with the aid of the Noether charge technique. Therefore, analyzing \mathcal{S} as the horizons become closer and closer each other and eventually merge, one can learn about the tendency of changes and calculate the entropy of nonextremal black holes in the extremality limit.

The radial coordinates of the event and inner horizons of the Reissner-Nordström geometry are related in a simple way

$$r_+ r_- = e^2. \tag{49}$$

When this two horizons merge, one has a degenerate (extreme) configuration with $r_+ = r_- = |e|$. It should be noted, however, that in the quantum corrected case Eq. (49) is no longer valid. Indeed, although the zeroth-order equation gives the exact location of the event horizon the same is not true for its second root, say, r_c . The condition $r_+ = r_-$ may be treated as a constraint equation which can be used to relate the electric charge and the exact location of the degenerate horizon. Now, assuming that the radius of the event horizon can be expanded as

$$r_+ = r_0 + \varepsilon r_1 + \mathcal{O}(\varepsilon^2), \tag{50}$$

where we do not ascribe any particular meaning to r_0 and r_1 , one has

$$r_+ = |e| - \frac{\varepsilon}{24\pi m^2 |e|^3} \left(4\alpha_4^{(s)} + 8\alpha_5^{(s)} + 3\alpha_6^{(s)} + 3\alpha_7^{(s)} + 6\alpha_8^{(s)} + 12\alpha_9^{(s)} \right). \quad (51)$$

Similarly, one can easily explore the consequences of vanishing of the surface gravity (temperature). Since the temperature as given by Eq.(46) is defined at the event horizon, we have a system of two equations, the first of which, $f(r_+) = 0$, is satisfied automatically whereas the solution of the second one

$$\frac{1}{r_+} - \frac{e^2}{r_+^3} + 8\pi\varepsilon r_+ \langle T_t^{(s)t}(r_+) \rangle = 0 \quad (52)$$

gives the desired result.

Now, taking the extremality limit in the general expression (34), after massive simplifications, one obtains

$$\mathcal{S} = \pi e^2 + \mathcal{O}(\varepsilon^2). \quad (53)$$

This result holds for any spin of the massive field and to required order it coincides with the analogous result calculated for the classical Reissner-Nordström black hole. Although the stress-energy tensor of the quantized field is known in the one loop approximation only, it is possible to construct \mathcal{S} of the nonextreme black hole up to the terms proportional to ε^2 . Calculations of the entropy of the extreme configuration to the second order would require knowledge of the stress-energy tensor beyond the one loop approximation.

The thus obtained \mathcal{S} is referred to as the entropy of the classical black hole or macroscopic entropy. The former designation is somewhat misleading in the present context as our black hole solution is, in fact, semiclassical. It should be noted, however, that the effective action of the massive quantized field is constructed from curvature, and the type of the field influences only the numerical coefficients that stand in front of purely geometric terms in (12). As such, it may be treated in the calculations as the classical higher derivative action functional.

The entropy of the extremal black hole as given by Eq. (53) is nonzero, and, therefore, it contradicts the Nernst formulation of the third law of thermodynamics, which asserts that the entropy of the system must go to zero or a universal constant as its temperature goes to zero. However, the subtle point is that the Nernst formulation should not be considered as a fundamental law of thermodynamics. Indeed, Wald in Ref [57] constructed some explicit examples that violate the Nernst law.

V. FINAL REMARKS

Now we are in position to compare our results with the results existing in literature. The $e = 0$ case has been briefly discussed at the end of the section IV A. On the other hand, one can treat the results of Lu and Wise [58] and $D = 4$ limit of the entropy calculated in Ref. [59] as special cases of present calculations. Indeed, observe that there are similarities between S of the quantized massive fields in a large mass limit and the most general effective action involving all (time-reversal invariant) curvature terms of dimension six considered by Lu and Wise. Setting $\alpha_1^{(s)} = \alpha_2^{(s)} = 0$, making use of the identity

$$R_{ab}^c{}^d R_{ef}^a{}^b R_{cd}^e{}^f - R_{af}^{cd} R_{ce}^{ab} R_{bd}^{ef} = \frac{1}{4} R_{ab}^{cd} R_{cd}^{ef} R_{ef}^{ab} \quad (54)$$

in $S_q^{(s)}$, and, subsequently, substituting

$$\frac{1}{192\pi^2 m^2} \alpha_i^{(s)} \rightarrow \alpha_i, \quad (55)$$

and absorbing the right hand side of Eq. (54) by the α_9 term, one obtains precisely the action considered in Ref. [58] up to natural typographical differences. Lu and Wise concentrated on the influence of the sixth-order terms on the Schwarzschild geometry. However, it is an easy exercise to generalize their results to the case of electrically charged black hole. Since it can be easily done by equating α_1 and α_2 to zero and simple rearrangement of the terms in Eq. (34) we shall not display the final result here.

To construct the entropy of the quantum-corrected black hole making use of Eq. (27) one has to calculate the functional derivatives of the Lagrangian with respect to the Riemann tensor and its symmetrized covariant derivatives. In this regard the (quantum-corrected) Reissner-Nordström geometry provides more sensitive test than the Schwarzschild solution. Indeed, in the Schwarzschild geometry $\alpha_1^{(s)} R \nabla_a \nabla^a R$ and $\alpha_2^{(s)} R_{ab} \nabla_c \nabla^c R^{ab}$ do not contribute to the entropy whereas in the Reissner-Nordström case one has nonvanishing contribution of $\alpha_2^{(s)}$ term. It would be interesting to analyze the black hole solutions with classical fields for which both terms do not vanish. This group of problem is under active considerations and the results will be published elsewhere.

APPENDIX

In this appendix we collect solutions to the semi-classical equations describing quantum-corrected Reissner-Nordström black hole. The general solution is too lengthy to be reproduced here. Explicit results for massive scalar, spinor and vector fields read [28, 31]:

$$f(r) = 1 - \frac{r_+}{r} + \frac{e^2}{r^2} - \frac{e^2}{rr_+} + \frac{8\pi\varepsilon}{m^2} \left(A^{(s)}(r) + \xi B^{(s)}(r) \right), \quad (\text{A.1})$$

where

$$\begin{aligned} A^{(0)}(r) = & \frac{1153}{1960} \frac{e^4}{r^8} + \frac{5}{112} \frac{r_+^2}{r^6} + \frac{13}{280} \frac{e^2}{r^6} - \frac{1237}{30240} \frac{r_+^3}{r^7} - \frac{113}{30240} \frac{1}{rr_+^3} \\ & + \frac{2327}{11340} \frac{e^6}{r^{10}} - \frac{613}{1680} \frac{e^4 r_+}{r^9} - \frac{613}{1680} \frac{e^6}{r^9 r_+} - \frac{1237}{30240} \frac{e^6}{r^7 r_+^3} \\ & + \frac{877}{70560} \frac{e^2}{rr_+^5} - \frac{1069}{70560} \frac{e^4}{rr_+^7} + \frac{4169}{635040} \frac{e^6}{rr_+^9} - \frac{2549}{10080} \frac{e^2 r_+}{r^7} \\ & + \frac{5}{112} \frac{e^4}{r_+^2 r^6} - \frac{2549}{10080} \frac{e^4}{r_+ r^7} + \frac{1369}{7056} \frac{e^2 r_+^2}{r^8} + \frac{1369}{7056} \frac{e^6}{r^8 r_+^2} \end{aligned} \quad (\text{A.2})$$

$$\begin{aligned} A^{(1/2)}(r) = & \frac{3}{280} \frac{r_+^2}{r^6} - \frac{27}{140} \frac{e^2}{r^6} + \frac{3}{280} \frac{e^4}{r_+^2 r^6} - \frac{149}{15120} \frac{r_+^3}{r^7} + \frac{1723}{5040} \frac{e^2 r_+}{r^7} \\ & + \frac{1723}{5040} \frac{e^4}{r_+ r^7} - \frac{149}{15120} \frac{e^6}{r^7 r_+^3} - \frac{2729}{17640} \frac{e^2 r_+^2}{r^8} - \frac{1073}{1764} \frac{e^4}{r^8} \\ & - \frac{2729}{17640} \frac{e^6}{r^8 r_+^2} + \frac{2687}{10080} \frac{e^4 r_+}{r^9} + \frac{2687}{10080} \frac{e^6}{r^9 r_+} - \frac{1639}{15120} \frac{e^6}{r^{10}} \\ & + \frac{67}{11760} \frac{e^2}{rr_+^5} - \frac{13}{15120} \frac{1}{rr_+^3} - \frac{767}{70560} \frac{e^4}{rr_+^7} + \frac{451}{70560} \frac{e^6}{rr_+^9} \end{aligned} \quad (\text{A.3})$$

$$\begin{aligned} A^{(1)}(r) = & \frac{47849}{10080} \frac{e^2 r_+}{r^7} + \frac{47849}{10080} \frac{e^4}{r_+ r^7} - \frac{577}{280} \frac{e^2}{r^6} - \frac{37}{560} \frac{r_+^2}{r^6} - \frac{37}{560} \frac{e^4}{r_+^2 r^6} \\ & + \frac{611}{10080} \frac{r_+^3}{r^7} + \frac{611}{10080} \frac{e^6}{r^7 r_+^3} - \frac{10393}{3920} \frac{e^2 r_+^2}{r^8} - \frac{35449}{3528} \frac{e^4}{r^8} \\ & - \frac{10393}{3920} \frac{e^6}{r^8 r_+^2} + \frac{26879}{5040} \frac{e^4 r_+}{r^9} + \frac{26879}{5040} \frac{e^6}{r^9 r_+} - \frac{31057}{11340} \frac{e^6}{r^{10}} \\ & - \frac{493}{14112} \frac{e^2}{rr_+^5} + \frac{11}{2016} \frac{1}{rr_+^3} + \frac{2393}{70560} \frac{e^4}{rr_+^7} - \frac{2389}{635040} \frac{e^6}{rr_+^9} \end{aligned} \quad (\text{A.4})$$

$$\begin{aligned} B^{(0)}(r) = & \frac{11}{60} \frac{r_+^3}{r^7} - \frac{1}{5} \frac{r_+^2}{r^6} - \frac{2e^2}{5r^6} - \frac{91}{90} \frac{e^6}{r^{10}} + \frac{1}{60} \frac{1}{rr_+^3} - \frac{29}{9} \frac{e^4}{r^8} + \frac{89}{60} \frac{e^4}{r_+ r^7} \\ & + \frac{89}{60} \frac{e^2 r_+}{r^7} + \frac{e^4}{18rr_+^7} - \frac{31}{30} \frac{e^6}{r^8 r_+^2} - \frac{31}{30} \frac{e^2 r_+^2}{r^8} + \frac{11}{60} \frac{e^6}{r^7 r_+^3} \\ & + \frac{113}{60} \frac{e^6}{r^9 r_+} - \frac{e^2}{20rr_+^5} - \frac{e^6}{45rr_+^9} + \frac{113}{60} \frac{e^4 r_+}{r^9} - \frac{e^4}{5r_+^2 r^6} \end{aligned} \quad (\text{A.5})$$

and

$$B^{(1/2)}(r) = B^{(1)}(r) = 0. \quad (\text{A.6})$$

For the function $\psi(r)$ one has

$$\begin{aligned}\psi^{(0)} = & \frac{\varepsilon}{\pi m^2} \left(-\frac{29}{1120} \frac{r_+^2}{r^6} - \frac{3}{80} \frac{e^2}{r^6} - \frac{29}{1120} \frac{e^4}{r_+^2 r^6} \right. \\ & + \frac{46}{441} \frac{e^2 r_+}{r^7} + \frac{46}{441} \frac{e^4}{r_+ r^7} - \frac{229}{1680} \frac{e^4}{r^8} \Big) \\ & + \frac{\varepsilon \xi}{\pi m^2} \left(\frac{7}{60} \frac{r_+^2}{r^6} - \frac{8}{15} \frac{e^2 r_+}{r^7} + \frac{7}{30} \frac{e^2}{r^6} \right. \\ & \left. - \frac{8}{15} \frac{e^4}{r_+ r^7} + \frac{13}{20} \frac{e^4}{r^8} + \frac{7}{60} \frac{e^4}{r_+^2 r^6} \right) \end{aligned} \quad (\text{A.7})$$

$$\begin{aligned}\psi^{(1/2)} = & \frac{\varepsilon}{\pi m^2} \left(-\frac{11}{1680} \frac{e^4}{r_+^2 r^6} - \frac{13}{245} \frac{e^4}{r_+ r^7} + \frac{37}{1120} \frac{e^4}{r^8} \right. \\ & \left. + \frac{7}{120} \frac{e^2}{r^6} - \frac{13}{245} \frac{e^2 r_+}{r^7} - \frac{11}{1680} \frac{r_+^2}{r^6} \right) \end{aligned} \quad (\text{A.8})$$

and

$$\begin{aligned}\psi^{(1)} = & \frac{\varepsilon}{\pi m^2} \left(\frac{131}{3360} \frac{e^4}{r_+^2 r^6} - \frac{2446}{2205} \frac{e^4}{r_+ r^7} + \frac{2141}{1680} \frac{e^4}{r^8} \right. \\ & \left. + \frac{173}{240} \frac{e^2}{r^6} - \frac{2446}{2205} \frac{e^2 r_+}{r^7} + \frac{131}{3360} \frac{r_+^2}{r^6} \right). \end{aligned} \quad (\text{A.9})$$

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